Preface

IFT has helped thousands of candidates successfully prepare for all three levels of the CFA® Program exam since 2011. IFT provides a complete learning system and preparation strategy. All IFT materials remain close to the CFA® Program curriculum. We help students develop a thorough understanding of fundamental concepts via active learning and exam practice. IFT Study Notes and Detailed Lecture Videos help you understand and retain curriculum concepts. Practice using IFT Quizzes and Question Bank is essential to test your grasp of the materials. Close to the exam date, IFT High-Yield Videos, High-Yield Notes and High-Yield Question Bank along with mock exams will help you achieve success.

Recommended Exam Preparation Strategy

The CFA Program curriculum is important and all IFT products focus on its core concepts. Start your preparation as early as possible. Give yourself at least 6 months for a thorough preparation and begin by making a schedule. For each reading in the curriculum, you should do the following:

1. Read IFT Study Notes for each reading. IFT recommends starting with Quantitative Methods. Do the questions at the end of each reading to help reinforce and retain the concepts.
2. If you need additional help, watch IFT Detailed Video Lectures. Pause, rewind, and review where you need more time to understand. Print out the slides and take your own notes as you are watching.
3. Read the examples in the curriculum (also known as blue-box examples) and do them on your own. Watch IFT example videos to further grasp concepts and practice them.
4. Do the practice problems from the curriculum at the end of each reading.
5. Do the IFT Question Bank for that reading.
6. After doing all the readings in a topic, take the IFT Topic Exam.
7. Two months prior to the exam or when you have done the above steps for every reading, you should start your revision using the IFT High-Yield Course.
8. One month before the exam you should attempt the CFA Institute and IFT mock exams. These should be done in a manner to replicate your real exam experience.

Below are some further details about IFT resources for Level I exam preparation.

IFT Detailed Lecture Videos: These videos closely follow the curriculum and cover all sections and learning outcomes of a reading. Lectures slides are available in PDF format. You can print these slides and make notes on them as you follow the video lecture. This practice is part of active learning whereby you make notes and carefully follow examples. The total duration of the videos is over 70 hours.

IFT Study Notes: IFT Study Notes are closely aligned with the Detailed Lecture Videos and present information in an easily understandable manner. The most important points, formulas, and examples are highlighted and explained. Reading the notes helps reinforce your understanding and grasp of concepts. Both notes and videos are organized to match the curriculum (reading number/section number) so you can stay close to it.

IFT Example Videos: These videos present examples and their worked solutions. They are similar but not exactly the same as the blue-box examples from the curriculum. The underlying concepts are the same, but the numbers might be different. The total duration of these videos is approximately 30 hours.

IFT Question Bank: Practicing lots of exam-like questions (and solving them correctly!) is the key to your exam success. The IFT Question Bank has over 3,000 questions with detailed explanations. Questions are available for every learning outcome, reading, and topic. The questions are arranged in tests of 20-25 questions each.

IFT Topic Exams: These will test your preparation at the topic level. A topic exam should be taken after you have studied all readings of a given topic and have completed practice questions for each reading. This is an essential step in preparing for the exam.

IFT Mock Exams: IFT has full length mock exams to get you exam-ready. Answers and detailed explanations are provided for self-grading.

IFT High-Yield Notes: IFT High-Yield Notes are based on Pareto's 80-20 rule according to which 80% of the exam questions are likely to be based on 20% of the curriculum. Hence these notes focus on the 20% material which is most testable. They summarize the most important concepts from each reading in two to five pages.

IFT High-Yield Videos: These are video lectures based on the High-Yield Notes. Each reading is covered in 10 to 20 minutes. Total duration is approximately 15 hours.

IFT High-Yield Q-Bank: The IFT High-Yield Q-Bank has between 600 and 700 questions covering concepts which are most likely to show up on the exam. All learning outcomes are covered and there is minimum overlap of the questions.

Q&A with IFT: When are you are stuck on a concept, example, or question, then the 'Q&A with IFT' service is where to turn to. Submit your query, and an IFT instructor will provide an individualized response.

Finally, be sure to follow IFT on social media and visit www.ift.world for my latest blogs and study advice. Do reach out to the IFT team if you need any help or have questions.

I wish you all the best in your studies.

Warm regards,

Arif Irfanullah, CFA

R06 Time Value of Money

Introductory Note

Financial Calculator: CFA Institute allows only two calculator models during the exam:

- Texas Instruments BA II Plus (including BA II Plus Professional) and
- Hewlett Packard 12C (including the HP 12C Platinum, 12C Platinum 25th anniversary edition, 12C 30th anniversary edition, and HP 12C Prestige)

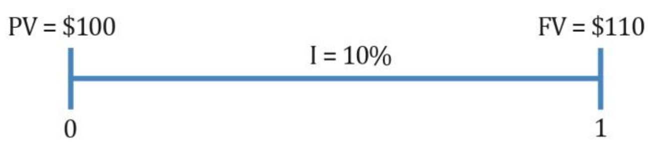
Unless you are already comfortable with the HP financial calculator, we recommend using the Texas Instruments financial calculator. Explanations and keystrokes in our study materials are based on the Texas Instruments BA II Plus calculator.
Before you start using the calculator to solve problems, we recommend that you set the number of decimal places to 'floating decimal'.

1. Introduction

If you have $100 today, versus an option to receive $100 after three years, what would you prefer?

Obviously, you would prefer $100 today. Even though you have the same amount ($100) in both cases, you prefer $100 today. This means that there has to be some value associated with time, because you are putting more value on the $100 that you are getting today, relative to the $100 at a later point in time. This is known as 'time value of money.'

Let us say that you are indifferent between $100 dollars today versus $ 110 after one year.



Present value (PV): The money today or the value today is called the present value (PV = 100). This could be an investment which you make at time 0.

Future value (FV): The value at a future point in time is called the future value (FV = 110).

Interest rate (I): The relationship or the link between present value and future value is established through an interest rate (I = 10%).

In this reading, we are essentially going to talk about these concepts: present value (PV), future value (FV), and the way we link these two concepts using interest rates (I).

2. Interest Rates: Interpretation

Let's discuss the different interpretations of interest rates using an example. Say you lend $900 today and receive $990 after one year (–ve sign indicates outflow).

Interest rates can be interpreted as:

1. Required rate of return: The fact that you are willing to give $900 today on the condition that you get $990 after one year means that to engage in this transaction, you require a return of 10%. (Simple calculation will show you that the interest rate in this transaction is 10%).
2. Discount rate: You can discount the money that you will receive after one year i.e. $990 at 10% to get the present value of $900 (990/1.1 = 900). Therefore, the 10% can also be thought of as a discount rate.
3. Opportunity cost: Let's say instead of lending the $900, you spent it on something else. You have then forgone the opportunity to earn 10% interest. Therefore, 10% can also be thought of as an opportunity cost.

Interest Rates: Investor Perspective

As an investor, we can think of the interest rate as a sum of the following components:

Interest rate = Real risk-free interest rate + Inflation premium + Default risk premium + Liquidity premium + Maturity premium

Let's look at the different components.

- Real risk-free interest rate: This is the rate that you get on a security that has no risk and is extremely liquid. We make an assumption here that there is no inflation.
- Inflation premium: We can then add on an inflation premium. Inflation premium is the expected annual inflation in the upcoming period.
- Default risk premium: We can also then add a default risk premium. This is the additional premium that investors require because of the risk of default.
  Example: Let's say that you lend $100 each to person A and person B. However, B has a high risk of default, so you are worried that he might not pay. Therefore you might demand a higher return from B as compared to A, because of the risk of default. This additional return that you demand is called the default risk premium.
- Liquidity premium: Next we have liquidity premium. This is the premium that investors demand because of the lack of liquidity of an investment.

Example: Think of two investments C and D which are similar in all regards. The only difference is that investment C is extremely liquid, whereas investment D is not that liquid. Clearly as investors, we will demand a higher return on D because it is not easy to sell. This additional return that we demand is called the liquidity premium.

- Maturity premium: Finally we have the maturity premium. This is the premium that investors demand on a security with long maturity. The maturity premium compensates investors for the increased sensitivity of the market value of debt to a change in market interest rates as maturity is extended.
  Example: Let's say we have two securities, E and F. Security E has a maturity of 1 year and security F has a maturity of 4 years. Because of the longer maturity, F has more risk, in terms of its price being more sensitive to changes in interest rate.
  Instructor's Note: You will understand this concept better when you study fixed income securities. But for now, you can take it as a given that F has higher risk because of the longer maturity.
  Obviously, investors will demand some compensation for the higher level of risk. This additional return that investors demand is called the maturity premium.

Nominal risk free rate:
Nominal risk-free rate = Real risk-free interest rate + Inflation premium.
So if the real risk-free rate is 3% and the inflation premium is 2%, then the nominal risk-free rate is 5%.

Instructor's Note: On the exam if you get a term 'risk-free rate' with no mention of whether the rate is real or nominal, then the assumption is that we are talking about the nominal risk-free rate.

Example

Investments	*Maturity (in years)*	*Liquidity*	*Default risk*	*Interest Rates(%)*
A	1	High	Low	2.0
B	1	Low	Low	2.5
C	2	Low	Low	r
D	3	High	Low	3.0
E	3	Low	High	4.0

1. Explain the difference between the interest rates on Investment A and Investment B.
2. Estimate the default risk premium.
3. Calculate upper and lower limits for the interest rate on Investment C, r.

Solution:

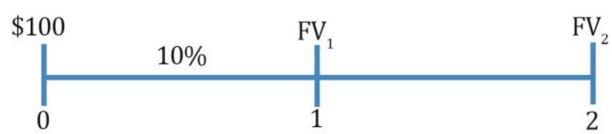
1. Investments A and B have the same maturity and the same default risk. However, B has a lower liquidity as compared to A. Hence, investors will demand a liquidity premium on B.

The difference between their interest rates i.e. 2.5 – 2.0 = 0.5% is equal to the liquidity premium.

2. Consider investments D and E, they have the same maturity, but different liquidity and different default risk. Let's make liquidity the same and create a new low liquidity version of D. This version will have a higher interest rate, because now investors will demand a liquidity premium. We have already determined that the liquidity premium is 0.5%. Therefore, the low liquidity version of D will have an interest rate of 3.0 + 0.5 = 3.5%.
   Now compare this version of D with investment E. The only difference between the two is default risk. E has a higher default risk. Therefore, the difference between their interest rates i.e. 4.0 – 3.5 = 0.5% must be equal to the default risk premium.
3. Notice that between B and C, the only difference is that C has a longer maturity. Therefore, interest rate of C must be higher than B (2.5%).
   Also notice that between C and the low liquidity version of D, the only difference is that C has a shorter maturity. Therefore, interest rate on C has to be lower than the low liquidity version of D (3.5%).
   So the range for C is 2.5 < r < 3.5.

3. The Future Value of a Single Cash Flow

Let's understand this concept with a simple example.
Say present value (PV) = \$100 and interest rate (r) = 10%.
What is the future value (FV) after one year?
What is the future value (FV) after two years?



The future value of a single cash flow can be computed using the following formula:

$$FV_N = PV\,(1 + r)^N$$

where:
FV_N = future value of the investment
N = number of periods
PV = present value of the investment
r = rate of interest

Therefore,

$$FV_1 = 100\,(1 + 0.1)^1 = \$110$$

$$FV_2 = 100\,(1 + 0.1)^2 = \$121$$

Notice that with compound interest, after two years we have $121. Whereas, with simple interest, after two years we would have $120. The difference between the two values ($1) represents the interest on interest component. In Year 2, we not only receive interest on the $100 principal, but we also receive interest on the $10 interest earned in Year 1 that has been reinvested.

Example

Cyndia Rojers deposits $5 million in her savings account. The account holders are entitled to a 5% interest. If Cyndia withdraws cash after 2.5 years, how much cash would she *most likely* be able to withdraw?

Solution:

$FV_N = PV(1 + r)^N$

$FV_{2.5} = 5(1 + 0.05)^{2.5} = \$5.649$ million

FV Calculation using a Financial Calculator

You will often use the following keys on your TI BA II Plus calculator:
N = number of periods
I/Y = rate per period
PV = present value
FV = future value
PMT = payment
CPT = compute

One important point to note is the signs used for PV and FV. If the value for PV is negative "-", then the value for FV is positive "+". An inflow is often represented as a positive number, while outflows are denoted by negative numbers.
Before you begin, set the number of decimal points on your calculator to 9 to increase accuracy.

Keystrokes	**Explanation**	**Display**
[2nd] [FORMAT] [ENTER]	Get into format mode	DEC = 9
[2nd] [QUIT]	Return to standard calc mode	0

Question: You invest $100 today at 10% compounded annually. How much will you have in 5 years?

The key strokes to compute the future value of a single cash flow are illustrated below.

Keystrokes	**Explanation**	**Display**
[2nd] [QUIT]	Return to standard calc mode	0
[2nd] [CLR TVM]	Clears TVM Worksheet	0

5 [N]	Five years/periods	N = 5
10 [I/Y]	Set interest rate	I/Y = 10
100 [PV]	Set present value	PV = 100
0 [PMT]	Set payment	PMT = 0
[CPT] [FV]	Compute future value	FV = -161.05

3.1. The Frequency of Compounding

When our compounding frequency is not annual, we use the following formula to compute future value:

$$FV_N = PV\left(1 + \frac{r_s}{m}\right)^{mN}$$

where:
r_s = the stated annual interest rate in decimal format
m = the number of compounding periods per year
N = the number of years

Let's understand this concept using an example.

You invest $80,000 in a 3-year certificate of deposit. This CD offers a stated annual interest rate of 10% compounded quarterly. How much will you have at the end of three years?

Solution:
There are two methods to solve this question.
<u>Formula Method</u>
PV is $80,000.
The stated annual rate is 10%.
The number of compounding periods per year is 4. The total number of periods is 4 x 3 = 12.
Therefore future value after 12 quarters (3 years) is

$$FV_{12} = \$80{,}000\ (1 + 0.025)^{12} = \$107{,}591$$

<u>Calculator Method</u>
You can also solve this problem using a financial calculator; the key strokes are given below:
N = 12, I/Y = 2.5%, PV = $80,000, PMT = 0, CPT FV = -$107,591
PMT is 0 because there are no intermediate payments in this example.

Example

Donald invested $3 million in an American bank that promises to pay 4% compounded daily. Which of the following is *closest* to the amount Donald receives at the end of the first year? Assume 365 days in a year.
A. $3.003 million
B. $3.122 million

C. $3.562 million

Solution
The correct answer is B.
Formula Method

$$FV_N = PV\left(1 + \frac{r_s}{m}\right)^{mN}$$

$$FV_{1\ year} = 3\ million\left(1 + \frac{0.04}{365}\right)^{365} = \$3.122\ million$$

Calculator Method
N = 365, I/Y = 4/365%, PV = $3 million, PMT = 0; CPT FV = -$3.122 million

3.2. Continuous Compounding

We saw examples with annual compounding. Then we discussed quarterly compounding and in the above example we looked at daily compounding. If we keep increasing the number of compounding periods until we have infinite number of compounding periods per year, then we can say that we have continuous compounding.

The formula for computing future values with continuous compounding is:

$$FV_N = PVe^{rN}$$

where:
r = continuously compounded rate
N = the number of years

Let's look at an example.

An investment worth $50,000 earns interest that is compounded continuously. The stated annual interest is 3.6%. What is the future value of the investment after 3 years?

Solution:

PV = $50,000; r = 0.036; N = 3

$$FV = 50{,}000\ e^{0.036 \times 3} = \$55{,}702$$

Concept Building Exercise

Assume that the stated annual interest rate is 12%. What is the future value of $100 at different compounding frequencies? What is the return?

Frequency	**Future value of $100**	**Return**
Annual	112.00	12.00%
Semiannual	112.36	12.36%
Quarterly	112.55	12.55%
Monthly	112.68	12.68%
Daily	112.75	12.75%
Continuous	112.75	12.75%

Instructor's Note:

1. For the same stated annual rate, the returns keep getting better as we compound more often.

2. If you have two banks that offer the following rates
A: 12.5% compounded annually
B: 12% compounded daily
Which offer is better?

Even though A's 12.5% looks better, B's 12% compounded daily will effectively give you a return of 12.75%. Therefore the offer from bank B is better.

3.3. Stated and Effective Rates

Now we come to the related concept of stated versus effective rates. In the above concept-building exercise, the stated rate was 12% across the board, but the effective rate that an investor actually earns depends on the compounding frequency. The effective rates were different for different compounding frequencies.

If we are given a compounding frequency, we can compute effective rates using the following formulae:

Effective annual rate for discrete compounding:

$$\text{EAR} = (1 + \text{periodic interest rate})^{m} - 1$$

where:
m = number of compounding periods in one year

- For daily compounding, m = 365
- For monthly compounding, m = 12
- For quarterly compounding, m = 4
- For semiannual compounding, m = 2

For example, for a stated annual rate of 12% and quarterly compounding, the EAR will be equal to:

$$\text{EAR} = (1 + 0.12/4)^4 - 1 = 1.1255 - 1 = 0.1255 = 12.55\%$$

Instructor's Note:
A lot of people get confused about the -1 at the end of the formula. The idea is actually fairly straight forward. Basically we have 1.03^4 which is telling us how much $1 will become at the end of 4 periods. $1 is going to become $1.1255. But this is not a rate. To figure out the rate we have to subtract the original $1 that we invested. So we are left with 0.1255 which is our effective rate.

Effective Annual Rate for continuous compounding:

$$\text{EAR} = e^{r_s} - 1$$

where r_s = stated annual interest rate

For example, for a stated annual rate of 12% and continuous compounding, the EAR will be equal to:

EAR = $e^{0.12}$ – 1 = 1.1275 – 1 = 0.1275 = 12.75%

4. The Future Value of a Series of Cash Flows

The different types of cash flows based on the time periods over which they occur include:

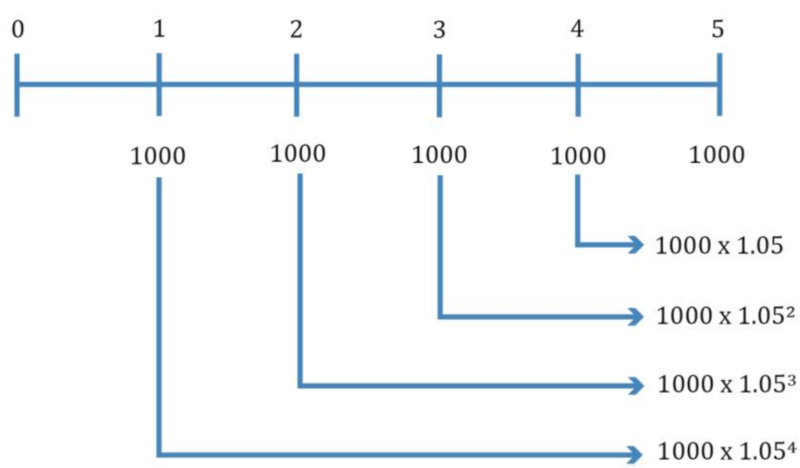
- **Annuity**: A finite set of equal sequential cash flows. The two types are
  - **Ordinary annuity**: The first cash flow occurs one period from now (indexed at t = 1).
  - **Annuity due**: The first cash flow occurs immediately (indexed at t = 0).
- **Perpetuity**: A set of equal never-ending sequential cash flows with the first cash flow occurring one period from today. (The period is finite in case of an annuity whereas in perpetuity it is infinite.)

4.1. Equal Cash Flows – Ordinary Annuity

Let's say that we have an ordinary annuity with A = \$1,000, r = 5% and N = 5, i.e. we are going to receive a payment of \$1,000 at the end of each year for the next 5 years and our discount rate is 5%. We can compute the FV of this annuity using the following three methods:

Brute-Force Method

We can take each individual cash flow and see how much each of these will be worth at the end of five years. Then we can add all the values to compute the total FV of the annuity.



For instance, the first \$1,000 deposit made at t = 1 will compound over four periods; the second deposit of \$1,000 will compound over three periods and so on. We then add the future values of all payments to arrive at the future value of the annuity which is \$5,525.63.

Formula Method

The future value of an annuity can also be computed using the following formula:

$$FV_N = A\left[\frac{(1+r)^N - 1}{r}\right]$$

where:
A = annuity amount
N = number of years

The term in square brackets is known as the 'future value annuity factor' (FVAF). This factor gives the future value of an ordinary annuity of $1 per period. Hence the formula given above can also be written as: FV = A x FVAF.

Therefore, using the formula:

$$FV_5 = 1{,}000\left[\frac{(1.05)^5 - 1}{0.05}\right] = \$5{,}525.63$$

Calculator Method

Given below are the keystrokes for computing the future value of an ordinary annuity.

Keystrokes	**Explanation**	**Display**
[2nd] [QUIT]	Return to standard calc mode	0
[2nd] [CLR TVM]	Clears TVM Worksheet	0
5 [N]	Five years/periods	N = 5
5 [I/Y]	Set interest rate	I/Y = 5
0 [PV]	0 because there is no initial investment	PV = 0
1000 [PMT]	Set annuity payment	PMT = 1000
[CPT] [FV]	Compute future value	FV = -5525.63

On the exam you should use the calculator method, because this is the fastest method and does not require you to memorize the annuity formula.

Example

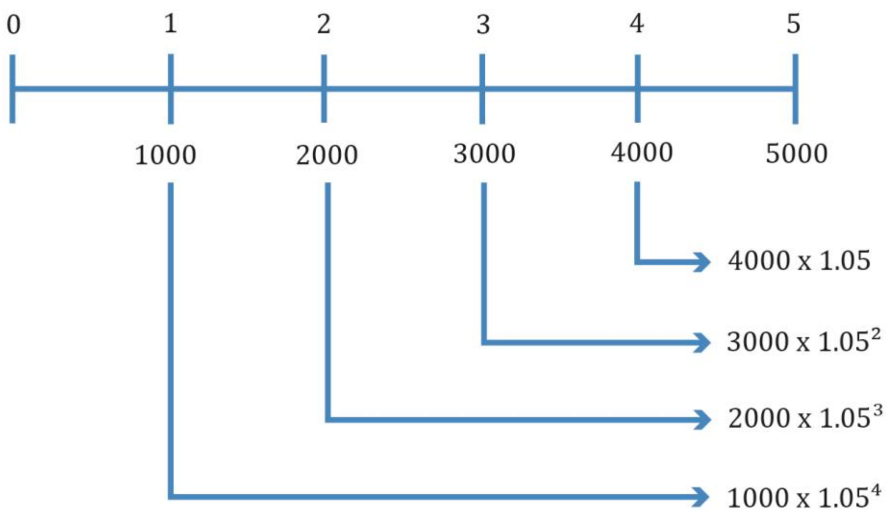
Haley deposits $24,000 in her bank account at the end of every year. The account earns 12% per annum. If she continues this practice, how much money will she have at the end of 15 years?

Solution:
N = 15, I/Y = 12%, PV = 0, PMT = $24,000; CPT → FV = -$894,713.15

4.2. Unequal Cash Flows

If cash flow streams are unequal, the future value annuity factor cannot be used. In this case, we find the future value of a series of unequal cash flows by compounding the cash flows one at a time.

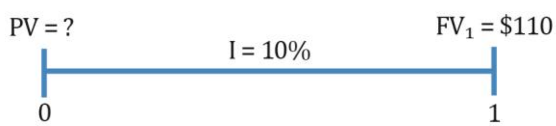
This concept is illustrated in the figure below. We need to find the future value of five cash flows: $1,000 at the end of Year 1; $2,000 at the end of Year 2; $3,000 at the end of Year 3; $4,000 at the end of Year 4; and $5,000 at the end of Year 5.



The future value is $5,000 + $4,000 x 1.05 + $3,000 x 1.05^2 +$ 2,000 x 1.05^3 + $1,000 x 1.05^4 = $16,038.

5. The Present Value of a Single Cash Flow

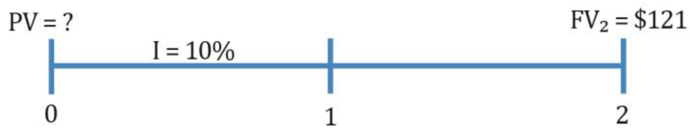
Let's say that one year from today you will receive a cash flow of $110. What is the value of that $110 today? (Assume that the interest rate is 10%)



$$\text{PV} = \frac{\text{FV}_1}{(1+0.1)^1} = \frac{\$110}{(1+0.1)^1} = \$100$$

The $110 one year from today has a present value of $100. In other words, you will be indifferent between $100 today and $110 one year from today.

What if you were going to receive $121 at the end of two years, what is its present value?



$$PV = \frac{FV_2}{(1+0.1)^2} = \frac{\$121}{(1+0.1)^2} = \$100$$

Using these two examples we can write a general formula for computing PV. Given a cash flow that is to be received in N periods and an interest rate of r per period, the PV can be computed as:

$$PV = \frac{FV_N}{(1+r)^N}$$

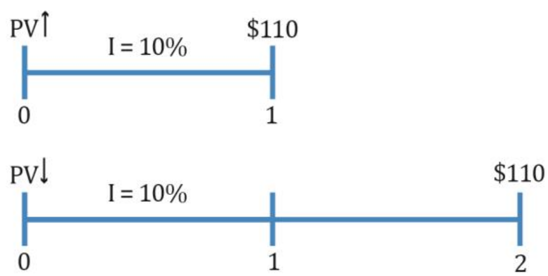
where:
N = number of periods
r = rate of interest
FV = future value of investment

Instructor's Note
Notice that this formula can also be obtained by simply rearranging the formula for FV that we studied earlier.

$FV_N = PV\,(1 + r)^N \rightarrow PV = \frac{FV_N}{(1+r)^N}$

Based on this formula we can make two observations

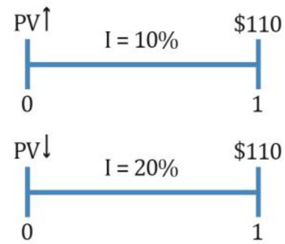
1. For a given discount rate, the farther in the future the amount to be received, the smaller the amount's present value.

   Mathematical explanation: In the first case the PV is \$110/1.1, whereas in the second case PV is $\$110/1.1^2$. Since we are dividing by a larger number the PV will be lower in the second case.

   Intuitive explanation: Clearly receiving a certain amount of money sooner is better than receiving the same amount of money latter.

2. Holding time constant, the larger the discount rate, the smaller the present value of a future amount.



In the first case PV is $110/1.1, whereas in the second case PV is $110/1.2. Clearly the PV is going to be lower in the second case, because we are dividing by a larger number.

Example

Liam purchases a contract from an insurance company. The contract promises to pay $600,000 after 8 years with a 5% return. What amount of money should Liam most likely invest? Solve using the formula and TVM functions on the calculator.

Solution:

Formula Method

$$PV = \frac{FV_N}{(1+r)^N} = \frac{\$600{,}000}{(1.05)^8} = \$406{,}104$$

Calculator Method

N = 8, I/Y = 5%, PMT = 0, FV = $600,000; CPT PV →PV = - $406,104

Example

Mathews wishes to fund his son, Nathan's, college tuition fee. He purchases a security that will pay $1,000,000 in 12 years. Nathan's college begins 3 years from now. Given that the discount rate is 7.5%, what is the security's value at the time of Nathan's admission?

Solution:

$$PV_3 = \frac{\$1{,}000{,}000}{(1.075)^9} = \$521{,}583.47$$

Example

Orlando is a manager at an Australian pension fund. 5 years from today he wants a lump sum amount of AUD40, 000. Given that the current interest rate is 4% a year, compounded monthly, how much should Orlando invest today?

Solution:

We have monthly compounding, therefore the inputs to our calculator will be
N = 5 x 12 = 60
I = 4 /12%
PMT = 0
FV = $40,000
CPT PV = - $32,760.12

6. The Present Value of a Series of Cash Flows

6.1. The Present Value of a Series of Equal Cash Flows

Ordinary Annuity

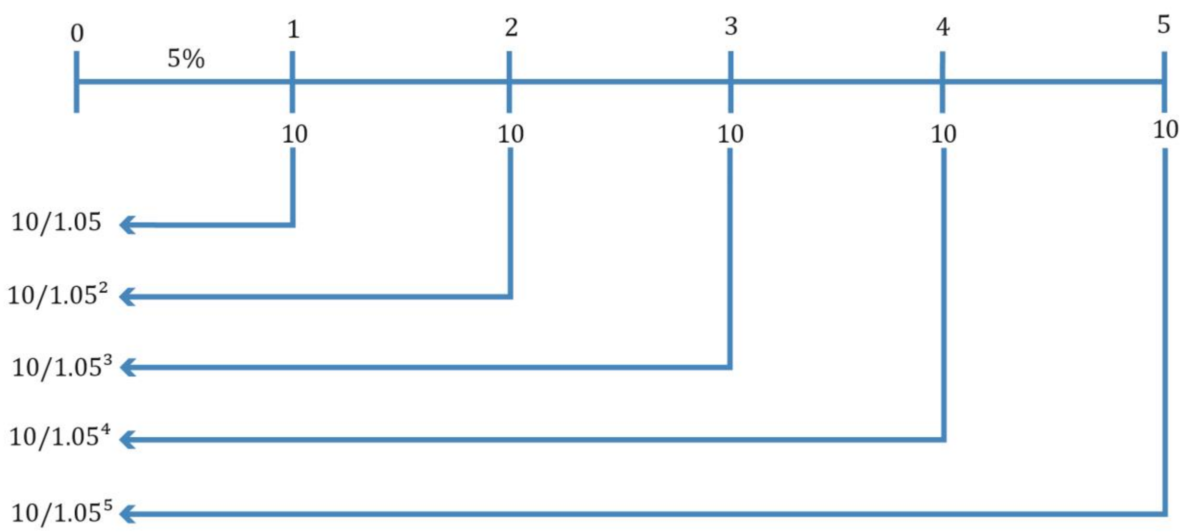
An ordinary annuity is a series of equal payments at equal intervals for a finite period of time. Examples of ordinary annuity: mortgage payments, pension income etc.

Consider an ordinary annuity with A = $10, r = 5% and N = 5.

We can compute the PV of this annuity using three methods

<u>Brute-Force Method</u>:

Simply take each cash flow and compute the PV. Add all values to get the PV for the annuity.



PV = $43.29.

<u>Formula Method</u>:

The present value can also be computed using the following formula:

$$PV = A\left[\frac{1-\left(\frac{1}{(1+r)^N}\right)}{r}\right]$$

where:
A = annuity amount
r = interest rate per period corresponding to the frequency of annuity payments
N = number of annuity payments
The term in square brackets is called the present value annuity factor (PVAF). Hence the equation above can also be written as: PV = A x PVAF.

Therefore, using the formula we get,

$$PV = 10\left[\frac{1-\left(\frac{1}{(1+0.05)^5}\right)}{0.05}\right] = \$43.29$$

Calculator Method:

The keystrokes to solve this using a financial calculator are given below:

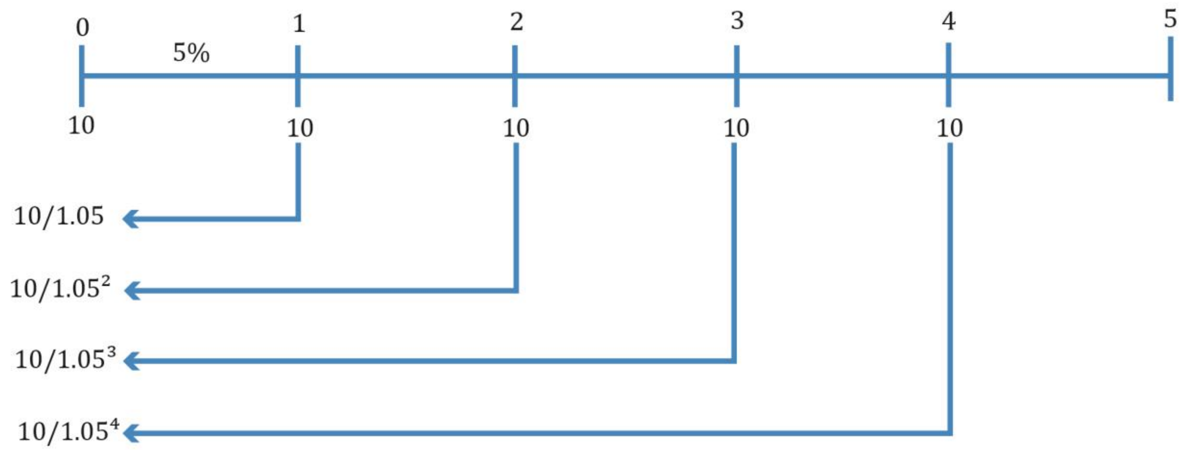
Keystrokes	*Explanation*	*Display*
[2nd] [QUIT]	Return to standard calc mode	0
[2nd] [CLR TVM]	Clears TVM Worksheet	0
5 [N]	Five years/periods	N = 5
5 [I/Y]	Set interest rate	I/Y = 5
0 [FV]	Set to 0 because there is no final payment other than the periodic annuity amounts	FV = 0
10 [PMT]	Set annuity payment	PMT = 10
[CPT] [PV]	Compute present value	PV = -43.29

Annuity Due
With an annuity due the first payment is received at the start of the first period. So if we have an annuity due with A = \$10, r = 5% and N = 5. The cash flows will be

Again, there are three methods to calculate the PV of this annuity due.

Brute-Force method

Take each cash flow and compute the PV. Add all values to get the PV for the annuity.



PV = $45.46.

Notice that with an annuity due you are receiving money faster, which means that the

PV $_{\text{annuity due}}$ ($45.46) > PV $_{\text{ordinary annuity}}$ ($43.29).

Formula method:

We can also use the following formula:

$$PV = A\left[\frac{1-\frac{1}{(1+r)^N}}{r}\right](1+r)$$

where:
A = annuity amount
r = interest rate per period corresponding to the frequency of annuity payments
N = number of annuity payments

Therefore using the formula,

$$PV = 10\left[\frac{1-\frac{1}{(1+0.05)^5}}{0.05}\right](1+0.05) = \$45.46$$

Instructor's Note:

Notice that $A\left[\frac{1-\frac{1}{(1+r)^N}}{r}\right]$ is basically the formula for computing the PV of an ordinary annuity.

If you use the ordinary annuity formula the PV that you get will be at time period -1. So this needs to be taken forward one period by multiplying it by (1 + r)

<u>Calculator Method</u>:

Set the calculator to BGN mode. This tells the calculator that payments happen at the start of every period. (The default calculator setting is END mode which means that payments happen at the end of every period). The keystrokes are shown below:

Keystrokes	**Explanation**	**Display**
[2nd] [BGN] [2nd] [SET]	Set payments to be received at beginning rather than end	BGN
[2nd] [QUIT]	Return to standard calc mode	BGN 0
[2nd] [CLR TVM]	Clears TVM Worksheet	BGN 0
5 [N]	Four years/periods	BGN N = 5
5 [I/Y]	Set interest rate	BGN I/Y = 5
10 [PMT]	Set payment	BGN PMT = 10
0 [FV]	Set future value	BGN FV = 0
[CPT] [PV]	Compute present value	BGN PV = -45.46
[2nd] [BGN] [2nd] [SET]	Set payments to be received at the end	END
[2nd] [QUIT]	Return to standard calc mode	0

Always remember to put your calculator back in the END mode after you are done with the calculations.

6.2. The Present Value of an Infinite Series of Equal Cash Flows – Perpetuity

A perpetuity is a series of never ending equal cash flows. The present value of perpetuity can be calculated by using the following formula:

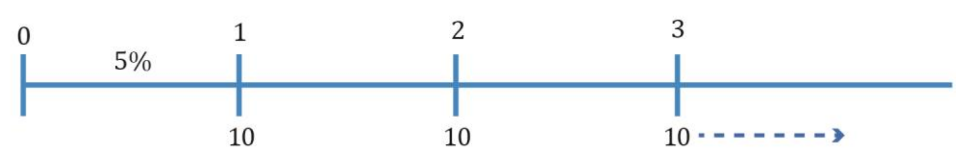
$$PV = \frac{A}{r}$$

where:

A = annuity amount

r = discount rate

Let's say that we have a really simple perpetuity where we receive $10 at the end of every year forever, and let's say that the interest rate is 5%.



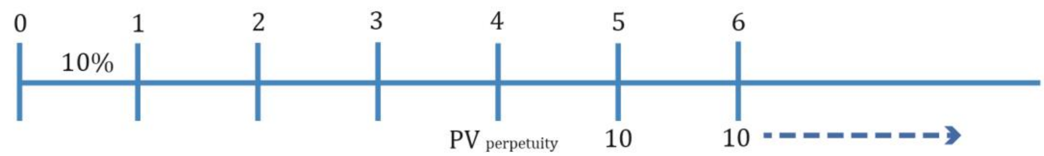
The PV of this perpetuity can be computed as:

$$PV = \frac{10}{0.05} = \$200$$

Instructor's Note:
Keep in mind that the present value of $200 is one period before the first cash flow. Many students show the present value of $200 at the same time as the first cash flow, which is incorrect.

6.3. Present Values Indexed at Times Other Than t=0

An annuity or perpetuity beginning sometime in the future can be expressed in present value terms one period prior to the first payment. That value can then be discounted back to today's present value.
Let's say you are offered a cash flow of $10 at the end of year 5, end of year 6, and so on forever. What is the PV of these cash flows, assuming a discount rate of 10%?



The PV of the perpetuity at the end of year 4 can be computed as:

$$PV_4 = \frac{10}{0.1} = \$100$$

This value has to be discounted back four periods to get the PV at time period 0.

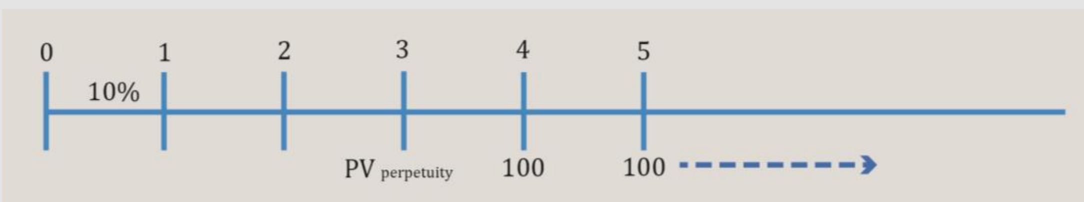
$PV_0 = \$100/1.1^4 = \$68.30$

Example

Bill Graham is willing to pay for a perpetual preferred stock that pays dividends worth $100 per year indefinitely. The first payment will be received at t = 4. Given that the required rate of return is 10%, how much should Mr. Graham pay today?

Solution:

The time line for this scenario is



The PV of the perpetuity at the end of year 3 can be computed as

$$PV_3 = \frac{100}{0.1} = \$1,000$$

This value has to be discounted back 3 periods to get the PV at time period 0.

PV_0 = \$1,000/1.1^3 = \$751.31

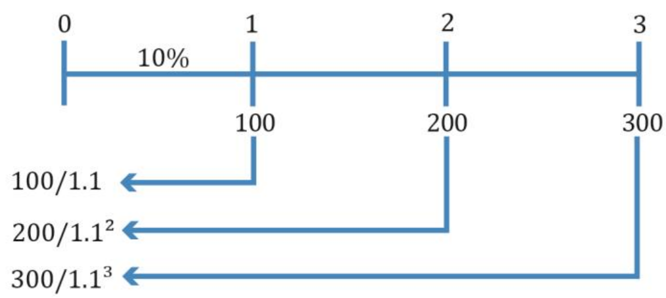
6.4. The Present Value of a Series of Unequal Cash Flows

When we have unequal cash flows, we can first find the present value of each individual cash flow and then sum the respective present values.

Let's say that we have the following cash flows:

Time period	*Cash Flow*
1	\$100
2	\$200
3	\$300

Assuming a discount rate of 10%, the PV of these cash flows can be computed as:



PV = 100/1.1 + 200/1.1^2 + 300/ 1.1^3 = \$481.59

Example

Andy makes an investment with the expected cash flow shown in the table below. Assuming a discount rate of 9% what is the present value of this investment?

Time Period	**Cash Flow(\$)**
1	50
2	100
3	150
4	200
5	250

Solution:

PV = 50/1.09 + 100/1.09^2 + 150/1.09^3 + 200/1.09^4 + 250/1.09^5 = \$550.03

We can also use the cash flow register on our financial calculator to solve this problem quickly. The key strokes are as follows:

Keystrokes	Explanation	Display
[2nd] [QUIT]	Return to standard mode	0
[CF] [2nd] [CLR WRK]	Clear CF Register	CF = 0
0 [ENTER]	No cash flow at t = 0	CF0 = 0
[↓] 50 [ENTER]	Enter CF at t = 1	C01 = 50
[↓] [↓] 100 [ENTER]	Enter CF at t = 2	C02 = 100
[↓] [↓] 150 [ENTER]	Enter CF at t = 3	C03 = 150
[↓] [↓] 200 [ENTER]	Enter CF at t = 4	C04 = 200
[↓] [↓] 250 [ENTER]	Enter CF at t = 5	C05 = 250
[↓] [NPV] [9] [ENTER]	Enter discount rate	I = 9
[↓] [CPT]	Compute NPV	550.03

7. Solving for Rates, Number of Periods, or Size of Annuity Payments

7.1. Solving for Interest Rates and Growth Rates

An interest rate can also be considered a growth rate. We can compute the rate using the formula method or the calculator method.

Example:
A $100 deposit today grows to $121 in 2 years. What is the interest rate? Use both the formula and the calculator method.
Using the formula $100(1+r)^2 = 121$. Therefore r = 0.1 or 10%
Using the calculator method, inputs to the calculator are
PV = -$100 (When we enter both PV and FV, they should be given opposite signs to avoid a calculator error.)
FV = $121
N = 2
PMT = 0
CPT I → I = 10%

Example:
The population of a small town is 100,000 on 1 Jan 2000. On 31 December 2001 the population is 121,000. What is the growth rate?
Inputs to the calculator are
PV = -$100,000
FV = $121,000
N = 2
PMT = 0
CPT I → I = 10%

Example:
You invest \$900 today and receive a \$100 coupon payment at the end of every year for 5 years. In addition, you receive \$1,000 and the end of year 5. What is the interest rate?
Inputs to the calculator are
PV = -\$900
FV = \$1,000
N = 5
PMT = 100
CPT I → I = 12.83%

7.2. Solving for the Number of Periods

Similarly, we can determine the number of periods given other information such as future value, present value and interest rate.

Example:
You invest \$2,500. How many years will it take to triple the amount given that the interest rate is 6% per annum compounded annually? Use both the formula and the calculator method.
Formula Method**:**

$FV = PV(1 + r)^N$

$7{,}500 = 2{,}500\,(1 + 0.06)^N$

$1.06^N = 3$

$N \times \ln 1.06 = \ln 3$

$N = \left(\frac{\ln 3}{\ln 1.06}\right) = 18.85$

Calculator Method**:**
Using the calculator: I/Y = 6%, PV = \$2,500, PMT = 0, FV = -\$7,500, CPT N = 18.85.

7.3. Solving for the Size of Annuity Payments

Given the number of periods, interest rate per period, present value, and future value, it is easy to solve for the annuity payment amount. This concept can be applied to mortgages and retirement planning. Consider the following example.

Example:
Freddie bought a car worth \$42,000 today. He was required to make a 15% down payment. The remainder was to be paid as a monthly payment over the next 12 months with the first payment due at t = 1. Given that the interest rate is 8% per annum compounded monthly, what is the approximate monthly payment?

Loan amount = 85% of \$42,000 = 0.85 x 42,000 = \$35,700
PV = \$35,700
N = 12
I/Y = 8/12%
FV = 0 CPT PMT → PMT = \$3,105.48

7.4. Review of Present and Future Value Equivalence

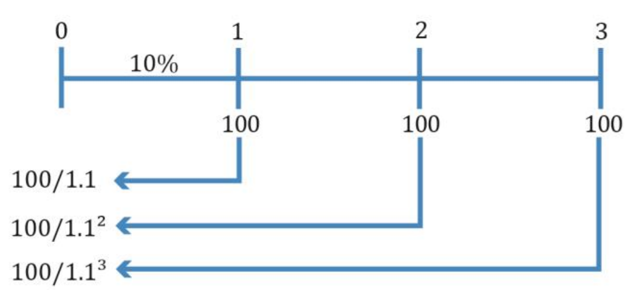
Let's say that we have an ordinary annuity with A = 10, r = 5% and N = 5.

As per our discussion so far, we can compute the PV and FV of this annuity
PV (at time 0) = $43.29 and FV (at time 5) = $55.26
According to the concept of present and future value equivalence, a lump sum of $43.29 at time 0 is equivalent to an annuity of $10 over five years. Further, both these options are equivalent to a lump sum of $55.26 at time 5. Given an interest rate of 5%, you would be indifferent between these choices.

7.5. The Cash Flow Additivity Principle

Amounts of money indexed at the same point in time are additive. For example, if you have the following cash flows:

You cannot simply add these three numbers. You have to take each of these numbers and bring them to a particular point in time. Let's say that we find the present values at time zero for each of these cash flows. According to this principle, these present values that are all indexed to time zero can be added.



Summary

LO.a: Interpret interest rates as required rates of return, discount rates, or opportunity costs.

An interest rate is the required rate of return. If you invest $100 today on the condition that you get $110 after one year, the required rate of return is 10%.
If the future value (FV) at the end of Year 1 is $110, you can discount at 10% to get the present value (PV) of $100. Hence, 10% can also be thought of as a discount rate.
Finally, if you spent $100 on taking your spouse out for dinner you gave up the opportunity to earn 10%. Thus, 10% can also be interpreted as an opportunity cost.

LO.b: Explain an interest rate as the sum of a real risk-free rate, and premiums that compensate investors for bearing distinct types of risk.

Interest rate = Real risk-free interest rate + Inflation premium + Default risk premium + Liquidity premium + Maturity premium.

Nominal risk free rate= real risk free rate + inflation premium

LO.c: Calculate and interpret the effective annual rate, given the stated annual interest rate and the frequency of compounding.

The stated annual interest rate is a quoted interest rate that does not account for compounding within the year. The effective annual rate (EAR) is the amount by which a unit of currency will grow in a year when we do consider compounding within the year.
Example: If the stated annual rate is 12% with monthly compounding, the periodic or monthly rate is 1%. Since $1 invested at the start of the year will grow to $1.01^{12} = 1.1268$, the EAR is 12.68%.

LO.d: Solve time value of money problems for different frequencies of compounding.

When our compounding frequency is not annual, we use the following formula to compute future value:

$$FV_N = PV\left(1 + \frac{r_s}{m}\right)^{mN}$$

where:
r_s = the stated annual interest rate in decimal format
m = the number of compounding periods per year
N = the number of years

If we keep increasing the number of compounding periods until we have infinite number of compounding periods per year, then we can say that we have continuous compounding. The formula to compute future value is:

$$FV_N = PVe^{rN}$$

where:

r = continuously compounded rate
N = the number of years

LO.e: Calculate and interpret the future value (FV) and present value (PV) of a single sum of money, an ordinary annuity, an annuity due, a perpetuity (PV only), and a series of unequal cash flows.

The future value and present value for a single sum of money can be calculated using the following formulae:

$FV = PV\ (1 + r)^N$ and $PV = FV/\ (1 + r)^N$

An **annuity** is a finite set of equal sequential cash flows occurring at equal intervals. There are two types of annuities:

- **Ordinary annuity**: Cash flows occur at the end of every period.
- **Annuity due**: Cash flows occur at the start of every period (hence, the Period 1 cash flow occurs immediately).

The future value of an ordinary annuity can be computed using the following formula:

$$FV_N = A\left[\frac{(1+r)^N - 1}{r}\right]$$

where:
A = annuity amount
N = number of years

The present value of ordinary annuity can be computed using the following formula:

$$PV = A\left[\frac{1-\left(\frac{1}{(1+r)^N}\right)}{r}\right]$$

where:
A = annuity amount
r= interest rate per period corresponding to the frequency of annuity payments
N = number of annuity payments

With an annuity due the first payment is received at the start of the first period. The formula to calculate present value of annuity due is as follows:

$$PV = A\left[\frac{1-\frac{1}{(1+r)^N}}{r}\right](1+r)$$

where:
A = annuity amount
r = interest rate per period corresponding to the frequency of annuity payments
N = number of annuity payments

Alternatively, you may also use the TVM keys on the calculator instead of the formulas to compute the present values and the future values of annuities.

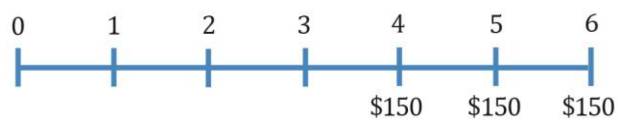
A perpetuity is a series of never ending equal cash flows. The present value of perpetuity can be calculated by using the following formula:

$$PV = \frac{A}{r}$$

where:
A = annuity amount
r = discount rate

LO.f: Demonstrate the use of a timeline in modeling and solving time value of money problems.

You can solve time value of money questions by showing cash flows on a timeline such as the one shown below:



Say you will receive \$150 at the end of Year 4, Year 5, and Year 6 and you want to calculate the PV at time 0. You can treat the three payments as an annuity and calculate the PV at the end of year 3. This value, assuming a 10% discount rate, is: \$373.03. We can then further discount \$373.03 to time 0. Plug: FV = \$373.03, N = 3, I = 10%, PMT = 0. Compute PV. You should get \$280.26.

Practice Questions

1. Interest rates can be *least likely* interpreted as:
   A. discount rates.
   B. opportunity costs.
   C. sunk costs.

2. The following information is provided regarding a security whose nominal interest rate is 10%:
   - The real risk-free rate of return is 4%
   - The default risk premium is 1%
   - The maturity risk premium is 1%
   - The liquidity risk premium is 2%

   An investor wants to determine the inflation premium in the security's return. The inflation premium is *closest* to:
   A. 2%.
   B. 4%.
   C. 6%.

3. Which of the following fixed income instruments has the *highest* effective annual rate (EAR)?

	Compounding frequency	**Annual interest rate**
Instrument 1	Monthly	6.20%
Instrument 2	Quarterly	6.25%
Instrument 3	Continuously	6.00%

   A. Instrument 1.
   B. Instrument 2.
   C. Instrument 3.

4. How much should be invested today at 8% interest compounded quarterly to accumulate $10,000 five years from today? The amount that must be invested today is *closest* to:
   A. $6,210.
   B. $6,730.
   C. $6,840.

5. The amount an investor will have in 10 years, if $1000 is invested today at a continuously compounded rate of 7%, will be *closest* to:
   A. $2,014.
   B. $2,038.
   C. $2,044.

6. Nancy Scott is buying a house. She expects her budget to allow a monthly payment of $2000 on a 20-year mortgage with a stated annual interest rate of 6 percent. If Ms. Scott puts a 15 percent down payment, the most she can pay for the house is *closest* to:
   A. $279,160.
   B. $328,425.
   C. $336,160.

7. A tenant pays rent of $800 monthly due on the first day of every month. If the annual interest rate is 7 percent, the present value of a full year's rent is *closest* to:
   A. $9,245.
   B. $9,300.
   C. $9,355.

8. The preferred shares of Crane Industries are expected to pay a $10 dividend forever, starting from the end of this year. If the required rate of return on equivalent investments is 9%, a share of Crane Industries preferred stock should be worth:
   A. $90.50.
   B. $111.10.
   C. $124.60.

9. An investment is expected to produce the cash flows of $100, $200, and $300 at the end of the next three years. If the required rate of return is 10%, the present value of this investment is closest to:
   A. $456.65.
   B. $475.83.
   C. $481.59.

10. James Miller wants to save for his son's college tuition. He will have to pay $40,000 at the end of each year for the four years that his son attends college. He has 6 years until his son starts college to save up for his tuition. Using a 8% interest rate compounded annually, the amount Miller would have to save each year for 6 years is *closest* to:
    A. $16,190.
    B. $18,060.
    C. $19,530.

Solutions

1. C is correct. Interest rates can be interpreted as
   - Required rate of return: It is the minimum rate that the investors require to make investments.
   - Discount rate: Future values can be discounted by interest rates to arrive at present values.
   - Opportunity costs: If instead of investing, you spend the money on something else, then you have given up the opportunity to earn interest.

   Sunk cost is a cost that has already been incurred and cannot be recovered.

2. A is correct. Nominal interest rate = real risk-free rate of return + inflation premium + risk premiums (default, liquidity, maturity premiums)
   Therefore,
   Inflation premium = 10% - 4% - 1% - 1% - 2% = 2%

3. B is correct. Use the EAR (effective annual rate) to compare the investments:

Instrument	Formula	EAR
Instrument 1	(1 + .062/12)^12 – 1	6.379%
Instrument 2	(1 + .0625/4)^4 – 1	6.398%
Instrument 3	e^(0.060 × 1) – 1	6.183%

4. B is correct. Enter into your financial calculator FV=10,000, N=5 x 4 = 20, I/Y =8/4 = 2, PMT=0, and solve for PV. PV = -6,729.71

5. A is correct. The future value of an amount calculated using continuous compounding is:
   $\text{FV} = \text{Pv} \times \text{e}^{\text{rt}}$
   Thus:
   $\text{FV} = 1000 \times \text{e}^{0.07\times 10} = \$2{,}013.75$

6. B is correct. The consumer's budget will support a monthly payment of $2,000. Given a 20-year mortgage at 6 percent, the loan amount will be $279,161 (N = 20 x 12 = 240, %I = 6/12 = 0.5, PMT = 2,000, FV = 0 solve for PV). If she makes a 15% down payment, then the most she can pay for the new house = $279,161/ (1 – 0.15) = $328,424.7.

7. B is correct. Using a financial calculator: PMT = 800, I=7/12 =0.583, n=12 Compute annuity due PV, PV = $9299.6 (Put the calculator in BGN mode for annuity due calculations)

8. B is correct.
   $\text{PV}_{\text{perpetuity}} = \frac{\text{PMT}}{\text{I/Y}} = \frac{\$10}{0.09} = \$111.11$

9. C is correct. Using your cash flow keys, (CF0=0), CF1 = 100, CF2 = 200, CF3= 300, I= 10
   Compute PV, PV = $481.59

10. B is correct. Using a financial calculator, we first need to calculate the total value of the tuition fees needed at the end of 6 years. Note that the first payment of 40,000 needs to be made 7 years from today.
    N = 4, I/Y = 8, PMT = 40,000, FV = 0
    Compute PV: PV= $132,485.07. This is the amount of money needed at the end of 6 years.

    Using $132,485.07 as the FV for the saving phase annuity, we compute yearly deposits with the inputs:
    N=6, I/Y = 8, PV = 0, FV = 132,485.07
    Compute PMT: PMT = $18059.75 ~ $18060.

