

2018 CFA[®] Exam Prep

IFT High-Yield Notes[®]

Quantitative Methods (Sample)

Level I

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Foreword

The **IFT High-Yield Course** is based on Pareto's 80-20 rule according to which 80% of the exam questions are likely to be based on 20% of the curriculum. Hence this course focuses on the 20% material which is most testable.

We call this the "High-Yield Course" because your investment (time and money) is low but the potential return (passing the exam) is high! As with high-yield investments in the world of finance, there is risk. Your exam will contain some questions which are not addressed in the High-Yield Course. However, we believe that such questions will be few and if you complement the High-Yield course with sufficient practice, the probability of passing the exam is high.

The IFT High-Yield course has three components: notes, lectures and questions.

1. **IFT High-Yield Notes**[®] summarize the most important concepts from each reading in 2 to 5 pages. Key formulas and facts are presented in blue boxes while examples appear in gray boxes.
2. **IFT High-Yield Lectures**[®] are online video lectures based on the notes. Each reading is covered in 10 to 20 minutes.
3. **High-Yield Q-Bank**[®] has between 600 and 700 questions covering concepts which are most likely to show up on the exam.

The High-Yield Course can be used on a 'stand-alone' basis if you are time-constrained. However, if you do have time, we recommend taking the High-Yield Course along with IFT's regular material. This will help ensure sufficient mastery of the entire curriculum.

Many candidates complain that they forget material covered earlier. The High-Yield Course addresses this problem by helping you to quickly revise key concepts.

Thank you for trusting IFT to help you with your exam preparation.

R6 Time Value of Money

Interpretation of interest rates

Interest rates can be interpreted as:

- **Required rate of return:** If you invest \$100 today on the condition that you get \$110 after a year, then your required rate of return is 10%.
- **Discount rate:** For the same example, if you discount the future cash flow of \$110 using a discount rate of 10%, you get a present value of \$100.
- **Opportunity cost:** If you spend the \$100 today instead of investing, then you lose the opportunity of earning 10% interest. In this sense interest rates can also be thought of as opportunity costs.

Components of interest rates

Interest rates have the following components:

- **Real risk-free rate:** Return on an investment with zero risk, assuming no inflation.
- **Inflation premium:** Extra return required to compensate for inflation.
- **Default risk premium:** Extra return required to compensate for the risk that the borrower will not make the promised payments.
- **Liquidity premium:** Extra return required to compensate for the risk of receiving less than the fair value of an investment if it must be sold quickly for cash.
- **Maturity premium:** The prices for longer-term bonds are more volatile than shorter-term bonds i.e. they have more maturity risk. Extra return required to compensate for maturity risk is called maturity premium.

Interest rate = real risk-free rate + inflation premium + default risk premium + liquidity premium + maturity premium.

Nominal interest rate = real risk-free rate + inflation premium

In a country ABC, the real risk-free rate is 4% and the expected inflation is 3%. Company X domiciled in this country issues a 5-year bond with an estimated default risk premium of 2%, liquidity premium of 1% and maturity premium of 1%. Calculate the interest rate of this bond.

Solution:

The interest rate for this bond will be $4 + 3 + 2 + 1 + 1 = 11\%$.

Effective annual rate

The stated annual rate is a quoted interest rate that does not consider the effect of compounding. The effective annual rate (EAR) is the rate at which money invested will grow in a year when we do consider compounding.

The EAR when there are m compounding periods in a year is:

$$\text{Effective annual rate} = \left(1 + \frac{\text{stated annual rate}}{m}\right)^m - 1$$

The EAR for continuous compounding is:

$$\text{Effective annual rate} = e^{\text{stated annual rate}} - 1$$

Calculate the EAR for:

1. A stated annual rate of 12% and semiannual compounding.
2. A stated annual rate of 12% and quarterly compounding.
3. A stated annual rate of 12% and monthly compounding.
4. A stated annual rate of 12% and continuous compounding.

Solution:

1. $\text{EAR} = 1.06^2 - 1 = 12.36\%$
2. $\text{EAR} = 1.03^4 - 1 = 12.55\%$
3. $\text{EAR} = 1.01^{12} - 1 = 12.68\%$
4. $\text{EAR} = e^{0.12} - 1 = 12.75\%$

Note that for the same stated annual rate, the EAR increases as the frequency of compounding increases.

Non-annual compounding frequencies

Step 1: Divide the stated annual interest rate by the number of compounding periods per year (m).

Step 2: Multiply the number of years by the number of compounding periods per year (m).

Step 3: Use the following formula to calculate future value.

$$\text{Future value} = \text{present value} \left(1 + \frac{\text{stated interest rate}}{m}\right)^{mN}$$

You invest \$10,000 in a 5-year bond. The bond offers a stated annual interest rate of 12% compounded semi-annually. What will be the value of the investment at the end of five years?

Solution:

Step 1: $12 / 2 = 6\%$

Step 2: $5 \times 2 = 10$ periods

Step 3: $\text{FV} = 10,000 (1.06)^{10} = \$17,908.47$

Calculating PV and FV of different cash flows

Present value is the current value of a future cash flow.

- Longer the time period till the future amount is received, lower the present value.
- Higher the discount rate, lower the present value.

Future value is the value to which an investment will grow after one or more compounding periods.

- Longer the time period till which the investment is allowed to grow, higher the future value.
- Higher the interest rate, the higher the future value.

The future value and the present value of a single sum of money can be calculated by using the formulae given below or by using the TVM keys on a financial calculator (recommended approach for the exams).

$$FV = PV (1 + I/Y)^N$$

$$PV = FV / (1 + I/Y)^N$$

You invest \$100 today at an interest rate of 10% for 5 years. How much will you receive after five years?

Solution:

Plug the following values in the calculator.

N = 5; I/Y = 10; PV = 100, PMT = 0; CPT FV = \$161.05

An **ordinary annuity** is series of finite but equal cash flows which occur at the end of each period.

How much should you invest today at an interest rate of 10% to receive \$100 at the end of each year for 5 years?

Using the calculator: N = 5; I/Y = 10; PMT = 100; FV = 0; CPT PV = \$379.08

An **annuity due** is a series of finite but equal cash flows which occur at the start of each period.

How much should you invest today at an interest rate of 10% to receive \$100 at the beginning of each year for 5 years?

Solution:

Put the calculator in BGN mode and plug the following values. (Remember to exit the BGN mode once you are done with your calculations.)

N = 5; I/Y = 10; PMT = 100, FV = 0; CPT PV = \$416.98

A **perpetuity** is a series of equal cash flows at regular intervals occurring forever. The present value of perpetuity can be calculated as:

$$PV \text{ of a perpetuity} = \frac{PMT}{I/Y}$$

How much should you invest today at an interest rate of 10% to receive \$100 at the end of each year forever?

Solution:

PV = 100/0.1 = \$1,000

The present (future) value of any series of cash flows is equal to the sum of the present (future) values of the individual cash flows.

What is the future value of the following series of cash flows, given an interest rate of 10%? \$1,000 at the end of year 1, \$2,000 at the end of year 2, \$3,000 at the end of year 3, \$4,000 at the end of year 4 and \$5,000 at the end of year 5.

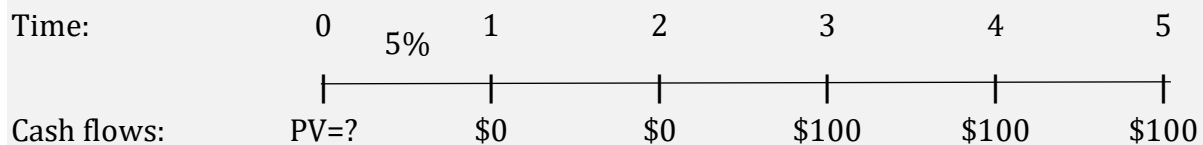
Solution:

The future value is $5000 + 4000 \times 1.1 + 3000 \times 1.1^2 + 2000 \times 1.1^3 + 1000 \times 1.1^4 = \$17,156$

Using time lines

A time-line is a diagram that shows the inflow and outflow of money at various time periods. Constructing a time-line will help you solve time value of money problems. Consider the following simple example.

Suppose you will receive \$100 at the end of Year 3, Year 4 and Year 5. What is the PV at time 0 given a discount rate of 5%?

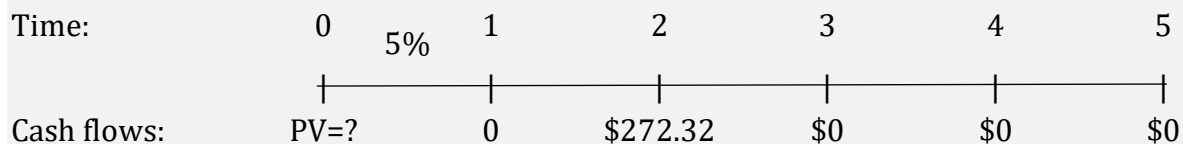
**Solution:**

To solve this problem you can draw the above time-line, depicting the cash flows. The three payments of \$100 can be treated as an ordinary annuity and we can calculate its PV at the end of year 2.

Plug the following values in the calculator:

$N = 3$; $I/Y = 5$; $PMT = \$100$, $FV = 0$; $CPT PV = \$272.32$

The time line now simplifies to:



To discount to time 0, plug the following values in the calculator:

$N = 2$; $I/Y = 5$; $PMT = 0$; $FV = 272.32$; $CPT PV = \$247$.

R7 Discounted Cash Flow Applications**Net present value (NPV) & Internal rate of return (IRR)**

The **NPV** of an investment is the present value of its cash inflows minus the present value of its cash outflows. The NPV of a project is calculated as:

$$NPV = CF_0 + \left[\frac{CF_1}{(1+r)^1} \right] + \left[\frac{CF_2}{(1+r)^2} \right] + \left[\frac{CF_3}{(1+r)^3} \right]$$

Consider a project which requires an initial investment of \$10,000. It is expected to generate \$5,000 in the first year, \$6,000 in the second year and \$7,000 in the third year.

The cost of capital for this project is 10%. Calculate the NPV of this project.

Solution:

$$NPV = -10,000 + \frac{5,000}{(1.1)^1} + \frac{6,000}{(1.1)^2} + \frac{7,000}{(1.1)^3}$$

$$NPV = \$4,763.33$$

Using the financial calculator: CF0 = -10,000; CF1 = 5,000; CF2 = 6,000; CF3 = 7,000; I = 10.
CPT NPV = 4,763.33

The **IRR** is the discount rate that makes the NPV equal to zero. i.e. it equates the PV of the cash inflows to the PV of the cash outflows. The IRR of a project is calculated as:

$$CF_0 = \left[\frac{CF_1}{(1+IRR)^1} \right] + \left[\frac{CF_2}{(1+IRR)^2} \right] + \left[\frac{CF_3}{(1+IRR)^3} \right]$$

Consider a project which requires an initial investment of \$10,000. It is expected to generate \$5,000 in the first year, \$6,000 in the second year and \$7,000 in the third year. Calculate the IRR of this project.

Solution:

$$10,000 = \left[\frac{5,000}{(1+IRR)^1} \right] + \left[\frac{6,000}{(1+IRR)^2} \right] + \left[\frac{7,000}{(1+IRR)^3} \right]$$

$$IRR = 33.87\%$$

Using the financial calculator: CF0 = -10,000; CF1 = 5,000; CF2 = 6,000; CF3 = 7,000; I = 10;
CPT IRR = 33.87

Conflict between NPV & IRR

NPV Rule

- For independent projects, accept a project if NPV > 0.
- For mutually exclusive projects, select the project with the highest positive NPV.

An **independent project** is one where the decision to accept or reject the project has no effect on any other projects being considered by the company. Hence, if a company is evaluating two independent projects, it can invest in both projects.

A **mutually exclusive project** is one where the acceptance of such a project will have an effect on the acceptance of another project. Hence, if a company is evaluating two mutually exclusive projects, then it can invest in only one of the projects.

IRR Rule

- For independent projects, accept if IRR > required rate of return.
- For mutually exclusive projects, select the project with the highest IRR.

Problems associated with IRR

- IRR wrongly assumes that the interim cash flows are reinvested at the IRR rate and not at the cost of capital.
- A project may have multiple IRRs or no IRR.

- For mutually exclusive projects, due to differences in project size or timing of cash flows, the IRR and NPV rankings may differ. If there is a conflict in ranking, go with the NPV rule.

Holding period return

The holding period return is the total return for holding an investment over a given time period. It is calculated as:

$$\text{HPR} = \frac{P_1 - P_0 + D_1}{P_0}$$

where:

P_0 = price at the start of the holding period

P_1 = price at the end of the holding period

D_1 = dividend/coupon paid by the investment at the end of the holding period

An investor bought a stock for \$100. Five months later, he received a dividend of \$4 and he sold the stock for \$108. Compute the HPR.

Solution:

$$\text{HPR} = \frac{108 - 100 + 4}{100} = 0.12 = 12\%$$

Money-weighted & Time weighted rate of return

Money-weighted rate of return

- The money-weighted rate of return is simply the IRR of a portfolio taking into account all cash inflows and outflows.
- If a manager controls the cash inflows and outflows of a portfolio, then use money-weighted return to measure performance.

An investor buys a stock for \$10 at time $t=0$. At the end of Year 1, he receives a dividend of \$1 and purchases another stock for \$12. At the end of Year 2, he receives a dividend of \$0.5 per share and sells both shares for \$13. Calculate the money-weighted return.

Solution:

Year	Outflow	Inflow	Net Cash Flow
0	\$10 to purchase the first share		-10
1	\$12 to purchase the second share	\$1 dividend received on first share	-11
2		\$1.00 dividend (\$0.50 x 2 shares) received \$26 received from selling 2 shares @ \$13 per share	+27

Enter the following in a calculator: CF0 = -10; CF1 = -11; CF2 = 27; CPT IRR = 18.28%. The

money weighted return is 18.28%.

Time-weighted rate of return

- Time-weighted rate of return is the compound growth rate at which \$1 invested in a portfolio grows over a given measurement period.
- If a manager cannot control the cash inflows and outflows of a portfolio, then use time-weighted return to measure performance.

An investor buys a stock for \$10 at time $t=0$. At the end of Year 1, he receives a dividend of \$1 and purchases another stock for \$12. At the end of Year 2, he receives a dividend of \$0.5 per share and sells both shares for \$13. Calculate the time-weighted rate of return.

Solution:

1. Break the measurement period into two sub-periods based on the timing of the cash flows.

Holding period 1	Beginning value = \$10 Dividends paid = \$1 Ending value = \$12
Holding period 2	Beginning value = \$24 (12 x 2) Dividends paid = \$1 (0.5 x 2) Ending value = \$26 (13 x 2)

2. Compute the HPY for each sub-period.

$$HPY_1 = (12 - 10 + 1)/10 = 30\%$$

$$HPY_2 = (26 - 24 + 1)/24 = 12.5\%$$

3. Calculate the compounded annual rate by taking the geometric mean of the two sub-periods.

$$(1 + TWRR)^2 = 1.30 \times 1.125; TWRR = 20.93\%$$

Yield measures for money market instruments

$$\text{Bank discount yield (BDY)} = \left(\frac{D}{F}\right) \times \left(\frac{360}{t}\right)$$

$$\text{Holding period yield (HPY)} = \frac{P_1 - P_0 + D_1}{P_0}$$

$$\text{Effective annual yield (EAY)} = (1 + HPY)^{\frac{365}{t}} - 1$$

$$\text{Money market yield (MMY)} = HPY \times \frac{360}{t}$$

Consider a T-Bill with a face value of \$100 and 60 days to maturity. It is selling at a discount of \$2 i.e. at a price of \$98. Calculate BDY, HPY, EAY and MMY.

Solution:

$$\text{Bank discount yield (BDY)} = \left(\frac{2}{100}\right) * \left(\frac{360}{60}\right) = 12\%$$

$$\text{Holding period yield (HPY)} = \frac{100 - 98 + 0}{98} = 2.04\%$$

$$\text{Effective annual yield (EAY)} = (1 + 0.0204)^{\frac{365}{60}} - 1 = 13.07\%$$

$$\text{Money market yield (MMY)} = 2.04\% \times \frac{360}{60} = 12.24\%$$

Conversion among yield measures

We can convert back and forth between holding period yields, money market yields, and effective annual yields by using the holding period yield, which is common to all the calculations. To calculate HPY use the following formulae. Then convert HPY to the required yield measure.

$$\text{HPY} = \text{MMY} \times \frac{t}{360}$$

$$\text{HPY} = (1 + \text{EAY})^{\frac{t}{365}} - 1$$

From BDY \rightarrow compute discount and initial price $P_0 \rightarrow$ compute HPY.

An investor purchased a \$100 T-bill that will mature in 60 days. The money market yield on the T-bill was 12.24% at the time of purchase. Compute HPY and EAY.

Solution:

$$\text{HPY} = 12.24 \times \frac{60}{360} = 2.04\%$$

$$\text{EAY} = (1 + 0.0204)^{\frac{365}{60}} - 1 = 13.07\%$$

An investor purchased a \$100 T-bill that will mature in 60 days. The bank discount yield on the T-bill was 12.00% at the time of purchase. Compute HPY and MMY.

Solution:

$$\text{BDY} = \left(\frac{D}{F}\right) * \left(\frac{360}{t}\right); 0.12 = \left(\frac{D}{100}\right) * \left(\frac{360}{60}\right); D = 2, P_0 = 98$$

$$\text{HPY} = \frac{P_1 - P_0 + D_1}{P_0} = \frac{100 - 98 + 0}{98} = 2.04\%$$

$$\text{MMY} = \text{HPY} \times \frac{360}{t} = 2.04\% \times \frac{360}{60} = 12.24\%$$

To calculate **bond-equivalent yield**, first convert to semiannual YTM, then use the following formula.

$$\text{Bond-equivalent yield} = 2 \times \text{semi-annual YTM}$$

A 3-month investment has a holding period yield of 2%. What is the yield on a bond equivalent basis?

Solution:

$$\text{First calculate the semiannual YTM} = (1 + 3\text{-month HPY})^2 - 1 = 1.02^2 - 1 = 4.04\%$$

$$\text{BEY} = 2 \times \text{semiannual YTM} = 2 \times 4.04 = 8.08\%$$

R8 Statistical Concepts and Market Returns

Fundamental concepts

Descriptive statistics refer to how large data sets can be summarized effectively to describe their important characteristics.

Inferential statistics refers to making forecasts, estimates or judgments about a large data set based on a small representative set.

Population includes all members of a particular group.

Sample is a subset drawn from a population.

Measurement Scales: Data can be measured using the following scales:

- **Nominal scales:** put data in categories but do not rank them.
- **Ordinal scales:** nominal scale + data can be ranked with respect to some characteristic.
- **Interval scales:** ordinal scale + the differences in the data values are meaningful.
- **Ratio scales:** interval scale + the ratios of value, such as twice or half as much are meaningful.

The scale on which data is measured determines the type of analysis that can be performed on the data.

Parameter, sample statistic & frequency distributions

Parameter is a descriptive measure of a population.

Sample statistic is a descriptive measure of a sample.

Frequency distribution is a tabular display of data categorized into a relatively small number of intervals or classes. It allows us to evaluate how data is distributed.

Frequency distribution of the marks scored by 100 students.

Marks Interval	Absolute Frequency
0 - 25	10
26 - 50	30
51 - 75	40
76 - 100	20

Relative frequencies and cumulative relative frequencies

Relative frequency is calculated as the absolute frequency of an interval divided by the total number of observations.

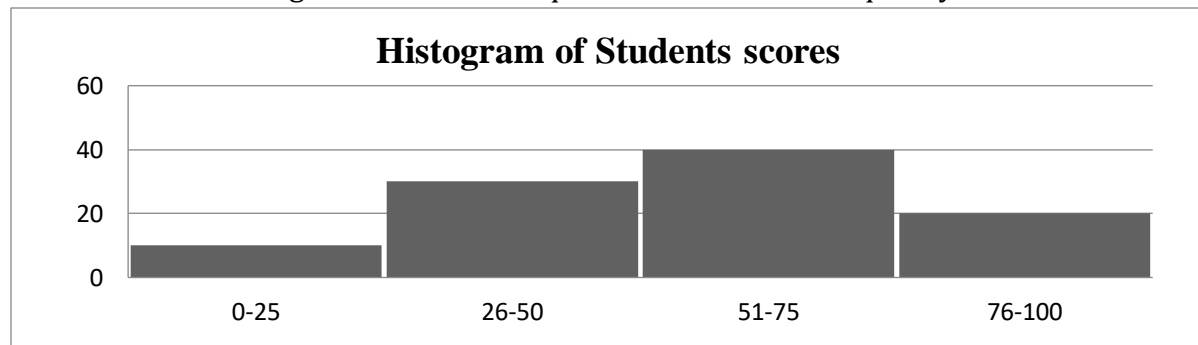
Cumulative relative frequency for an interval, is calculated as the sum of the relative frequencies of all intervals lower than and including that interval.

Frequency distribution of the marks scored by 100 students.

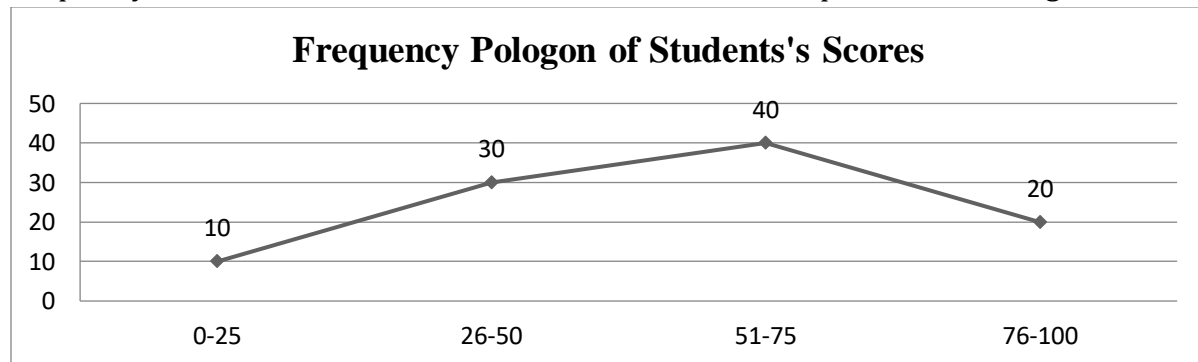
Marks Interval	Absolute Frequency	Relative Frequency	Cumulative Relative Frequency
0 - 25	10	10%	10%
26 - 50	30	30%	40%
51 - 75	40	40%	80%
76 - 100	20	20%	100%

Histograms & frequency polygon

Histogram is a bar chart of data that has been grouped together into a frequency distribution. The height of each bar is equal to the absolute frequency of each interval.



Frequency polygon plots the midpoints of each interval on the X-axis and the absolute frequency of that interval on the Y-axis, and connects these points with straight lines.



Measures of central tendency

Arithmetic mean is the sum of all the observations divided by the total number of observations. A population average is called population mean (μ). A sample average is called sample mean (\bar{x}). The sample mean is used as the 'best guess' approximation of the population mean.

$$\mu = \frac{\sum_{i=1}^N X_i}{N}$$

A stock had the following returns in the past three years: 10%, -5%, and 20%. Calculate the arithmetic mean.

Solution:

Arithmetic mean = $(10 - 5 + 20)/3 = 8.33\%$

Median is the midpoint of a data set that has been sorted from largest to smallest. If we have an even number of observations, then the median is the average of the two middle observations.

Calculate the medians for the following data sets:

A) 1, 2, 3, 4, 5

B) 1, 2, 3, 4, 5, 6

Solution:

A) Median = 3

B) Median = $(3+4)/2 = 3.5$

Mode is the value that occurs most frequently in a data set. A data set can have more than one mode, but only one mean and one median.

Calculate the mode for the following data set: 1, 2, 2, 3, 4, 4, 4, 5, 5

Solution:

Mode = 4

Geometric mean is used to calculate compound growth rate.

$$R_G = [(1 + R_1)(1 + R_2) \dots (1 + R_n)]^{1/n} - 1$$

A stock had the following returns in the past three years: 5%, -5%, and 20%. Calculate the geometric mean.

Solution:

$$R_G = [(1.05)(0.95)(1.20)]^{1/3} - 1 = 6.17\%$$

In a **weighted mean** different observations are given different weights as per their proportional influence on the mean.

$$\bar{X}_w = \sum_{i=1}^n w_i X_i$$

An investor has 20% of his portfolio in Stock A, 30% in Stock B and 50% in Stock C. If the returns were 4% on Stock A, 7% on Stock B and 8% on Stock C. Calculate portfolio return.

Solution:

$$\text{Portfolio return} = 0.2 \times 4 + 0.3 \times 7 + 0.5 \times 8 = 6.9\%$$

Harmonic mean is used to find average purchase price for equal periodic investments.

$$X_H = n / \sum_{i=1}^n \left(\frac{1}{X_i} \right)$$

An investor purchased \$1,000 worth of stock A each month for the past three months at prices of \$5, \$6 and \$7. Calculate the average purchase price of the stock.

Solution:

$$\text{Average purchase price} = 3 / (1/5 + 1/6 + 1/7) = 5.88$$

If returns are constant over time: $AM = GM = HM$

If returns are variable over time: $AM > GM > HM$

Quartiles, quintiles, deciles, & percentiles

A quantile is a value at or below which a stated fraction of the data lies. Some examples of quantiles include:

- **Quartiles:** distribution is divided into quarters
- **Quintiles:** distribution is divided into fifths.
- **Deciles:** distribution is divided into tenths.
- **Percentile:** distribution is divided into hundredths.

The formula for the position of a percentile in a data set with n observations sorted in ascending order is:

$$L_y = (n+1) y / 100$$

Calculate the first quartile of a distribution that consists of the following portfolio returns: 3%, 4%, 6%, 9%, 11%, 12%, 14%

Solution:

The first quartile = $(7+1) 25/100 = 2^{\text{nd}}$ item in the data set (4%). i.e. 25% of the observations lie below the second observations from the left.

Measures of dispersion

Range is the difference between the maximum and minimum values in a data set.

$$\text{Range} = \text{maximum value} - \text{minimum value}$$

The annual returns of a portfolio manager for the past 4 years are 4%, 2%, 6%, 8%. Calculate the range.

Solution:

$$\text{Range} = 8\% - 2\% = 6\%$$

Mean absolute deviation (MAD) is the average of the absolute values of deviations from the mean.

$$\text{MAD} = [\sum_{i=1}^n |X_i - \bar{X}|] / n$$

The annual returns of a portfolio manager for the past 4 years are 4%, 2%, 6%, 8%. Calculate MAD.

Solution:

$$X = (4 + 2 + 6 + 8) / 4 = 5$$

$$\text{MAD} = \frac{|4-5|+|2-5|+|6-5|+|8-5|}{4} = \frac{1+3+1+3}{4} = 2$$

Variance is defined as the mean of the squared deviations from the arithmetic mean.

$$\text{Population variance } \sigma^2 = \sum_{i=0}^N (X_i - \mu)^2 / N$$

$$\text{Sample variance } s^2 = \sum_{i=0}^n (X_i - \bar{X})^2 / (n - 1)$$

Standard deviation is the positive square root of the variance. It is often used as a measure of risk.

The annual returns of a portfolio manager for the past 4 years are 4%, 2%, 6%, 8%.

Calculate the population and sample standard deviations.

Solution:

It is advisable to use the financial calculator instead of the formula on the exams. The keystrokes for the above example are: [2nd] [DATA], [2nd] [CLR WRK], 4 [ENTER], [↓] [↓] 2 [ENTER], [↓] [↓] 6 [ENTER], [↓] [↓] 8 [ENTER], [2nd] [STAT] [ENTER], [2nd] [SET] press repeatedly till you see 1-V. Keep pressing [↓] to see the following values: N = 4, X = 5, Sx = 2.58, $\sigma_x = 2.24$.

Note: If you are asked to calculate the variances, simply square the standard deviations.

Chebyshev's inequality

According to Chebyshev's inequality, the proportion of the observations within k standard deviations of the arithmetic mean is at least:

$$1 - 1/k^2 \text{ for all } k > 1.$$

Determine the minimum percentage of observations in a data set that lie within 2 standard deviations from the mean.

Solution:

$$\% \text{ of population within 2 std deviations} = 1 - 1/k^2 = 1 - 1/2^2 = 75\%$$

Thus, Chebyshev's inequality permits us to make probabilistic statements about the proportion of observations within various intervals around the mean for any distribution with finite variance. As a result of Chebyshev's inequality, a two-standard deviation interval around the mean must contain at least 75 percent of the observations, no matter how the data is distributed.

Coefficient of variation & Sharpe ratio

Coefficient of variation measures the risk per unit of return. When evaluating investments, a lower value is better.

$$\text{CV} = \frac{s}{\bar{X}}$$

Calculate the coefficient of variation for the following portfolios and interpret the results.

	Mean return	Standard deviation
--	-------------	--------------------

Portfolio A	10%	9%
Portfolio B	6%	4%
Portfolio C	9%	2%

Solution:

$$CV_A = 9/10 = 0.9$$

$$CV_B = 4/6 = 0.66$$

$$CV_C = 2/9 = 0.22$$

Portfolio C has the lowest risk per unit of return, so it is the most attractive investment.

Sharpe ratio measures excess return per unit of risk. When evaluating investments, a higher value is better.

$$S_p = \frac{\bar{R}_p - \bar{R}_F}{s_p}$$

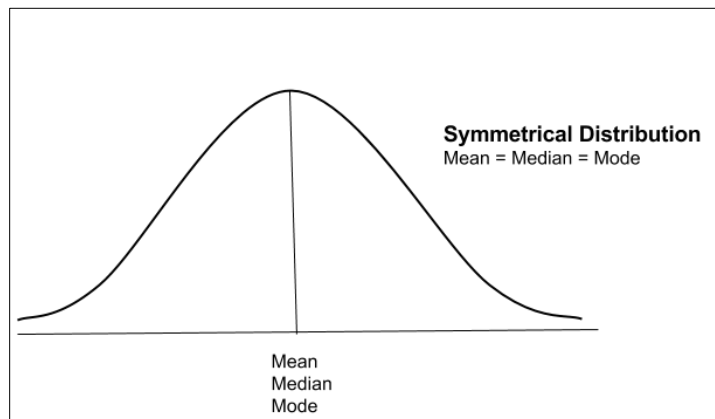
A portfolio has a mean return of 8% and a standard deviation of 10%. Calculate the Sharpe ratio, given that the risk-free rate is 3%.

Solution:

$$\text{Sharpe ratio} = (8 - 3)/10 = 0.5$$

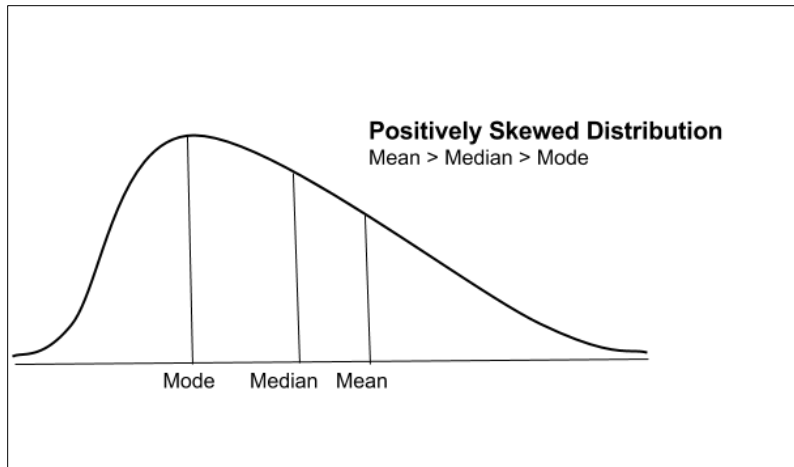
Symmetry and skewness in return distributions

A distribution is said to be **symmetrical** when the distribution on either side of the mean is a mirror image of the other. In a symmetrical distribution, mean = median = mode.

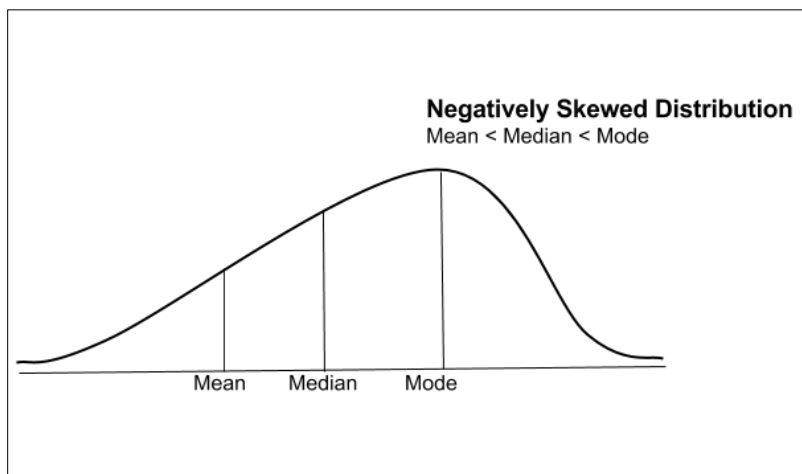


If a distribution is non-symmetrical, it is said to be skewed. Skewness can be negative or positive.

A **positively skewed distribution** has a long tail on the right side, which means that there will be frequent small losses and few large gains. Here the mean > median > mode. The extreme values affect the mean the most which is pulled to the right. They affect the mode the least.



A **negatively skewed distribution** has a long tail on the left side, which means that there will be frequent small gains and few large losses. Here the mean < median < mode. The extreme values affect the mean the most which is pulled to the left. They affect the mode the least.



Kurtosis

Kurtosis is a measure of the degree to which a distribution is more or less peaked than a normal distribution, which has a kurtosis of 3. Excess kurtosis = kurtosis - 3. An excess kurtosis with an absolute value greater than 1 is considered significant.

- A **leptokurtic** distribution is more peaked and has fatter tails than a normal distribution.
- A **platykurtic** distribution is less peaked and has thinner tails than a normal distribution.
- A **mesokurtic** distribution is identical to a normal distribution.

Using geometric and arithmetic means

The **geometric mean** is appropriate to measure past performance over multiple periods.

The portfolio returns for the past two years were 100% in year 1 and -50% in year 2. What was the mean return?

Solution:

Past return = geometric mean = $(2 \times 0.5)^{0.5} - 1 = 0\%$

The **arithmetic mean** is appropriate for forecasting single period returns.

Two possible returns for the next year are 100% and -50%. What is the expected return?

Solution:

Expected return = arithmetic mean = $(100 - 50)/2 = 25\%$

R9 Probability Concepts

Fundamental concepts

A **random variable** is an uncertain quantity/number.

An **outcome** is the observed value of a random variable.

An **event** can be a single outcome or a set of outcomes.

Mutually exclusive events are events that cannot happen at the same time.

Exhaustive events are those that include all possible outcomes.

- When you roll a die, the result is a random variable.
- If you roll a 2, it is an outcome.
- You can define an event as rolling a 2 or rolling an even number.
- Rolling a 2 and rolling a 3 are examples of mutually exclusive events. They cannot happen at the same time.
- Rolling an even number or Rolling an odd number are exhaustive events. They cover all possible outcomes.

Properties of probability

The two properties of probability are:

- The probability of any event has to be between 0 and 1.
- The sum of the probabilities of mutually exclusive and exhaustive events is equal to 1.

The methods of estimating probabilities are:

- Empirical probability: based on analyzing the frequency of an event's occurrence in the past.
- A priori probability: based on formal reasoning and inspection rather than personal judgment.

- Subjective probability: informed guess based on personal judgment.

Odds for and against the event

Odds for an event are defined as the probability of the event occurring to the probability of the event not occurring.

Odds against an event are defined as the probability of the event not occurring to the probability of the event occurring.

$$\text{Odds for } E = P(E) / [1 - P(E)]$$

$$\text{Odds against } E = [1 - P(E)]/P(E)$$

If the probability of the market rising tomorrow is 0.2. Calculate the odds for the market rising and against the market rising.

Solution:

Odds for the market rising = $0.2/0.8 = 1$ to 4

Odds against the market rising = $0.8/0.2 = 4$ to 1

Unconditional v/s conditional probabilities

Unconditional probability is the probability of an event occurring irrespective of the occurrence of other events. It is denoted as $P(A)$.

Conditional probability is the probability of an event occurring given that another event has occurred. It is denoted as $P(A|B)$, which is the probability of event A given that event B has occurred.

Multiplication, addition, and total probability rules

Multiplication rule is used to determine the joint probability of two events.

$$P(AB) = P(A|B) P(B)$$

For independent events, $P(A|B) = P(A)$. Hence the formula simplifies to

$$P(AB) = P(A) P(B)$$

We are given the following information:

- Probability that the economy will improve $P(E) = 0.6$
- Probability that stock price will increase given that the economy has improved $P(S|E) = 0.4$

Compute the probability that the stock price will increase **and** the economy will improve.

Solution:

$$P(SE) = P(S|E) \times P(E) = 0.4 \times 0.6 = 0.24$$

Addition rule is used to determine the probability that at least one of the events will occur.

$$P(A \text{ or } B) = P(A) + P(B) - P(AB)$$

For mutually exclusive events $P(AB) = 0$. Hence the formula simplifies to

$$P(A \text{ or } B) = P(A) + P(B)$$

We are given the following information:

- Probability that price of Stock A increases = $P(A) = 0.4$
- Probability that price of Stock B increases = $P(B) = 0.5$
- Probability that price of Stock A and Stock B increases = $P(AB) = 0.2$

Compute the probability that the price of stock A **or** the price of stock B increases.

Solution:

$$P(A \text{ or } B) = P(A) + P(B) - P(AB) = 0.4 + 0.5 - 0.2 = 0.7$$

Total probability rule is used to calculate the unconditional probability of an event, given conditional probabilities.

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \dots + P(A|B_n)P(B_n)$$

We are given the following information:

- Probability of above average economic growth = $P(B_1) = 0.4$
- Probability of average economic growth = $P(B_2) = 0.5$
- Probability of below average economic growth = $P(B_3) = 0.1$
- Probability of stock price rising given above average economic growth = $P(A|B_1) = 0.9$
- Probability of stock price rising given average economic growth = $P(A|B_2) = 0.6$
- Probability of stock price rising given below average economic growth = $P(A|B_3) = 0.1$

Compute the probability that the stock price will increase.

Solution:

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3) = 0.9 \times 0.4 + 0.6 \times 0.5 + 0.1 \times 0.1 = 0.67$$

Dependent v/s independent events

If the occurrence of one event does not influence the occurrence of the other event, then the events are called **independent events**. i.e. $P(A|B) = P(A)$ or $P(B|A) = P(B)$

If the probability of an event is affected by the occurrence of another event, then it is called a **dependent event**.

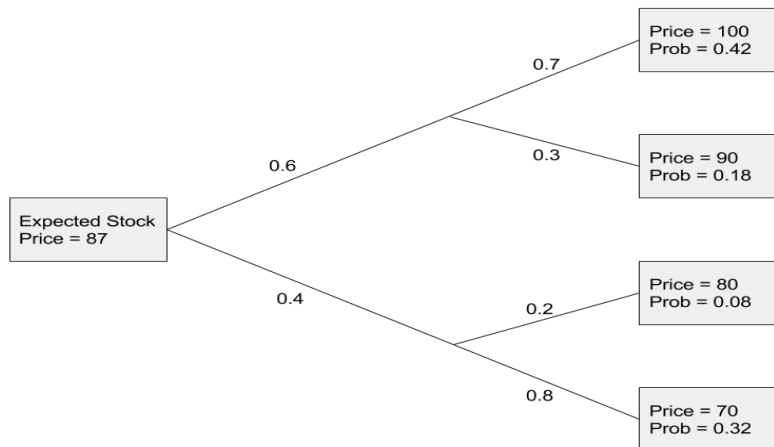
Using tree diagrams to represent an investment problem

A tree diagram helps plot the probabilities of various outcomes and depict expected values based on the paths chosen at each node.

There is a 0.6 probability of a good economy and a 0.4 probability of a poor economy. If there is a good economy, there is 0.70 probability that the stock price will be 100 and 0.3 probability that the stock price will be 90. If the economy is poor, there is a 0.2 probability that the stock price will be 80 and a 0.8 probability that the stock price will be 70. Calculate the expected stock price.

Solution:

Construct a tree diagram depicting the given scenario. The unconditional probability of stock price being 100 can be calculated by multiplying $P(\text{good economy}) \times P(\text{stock price being 100 when the economy is good}) = 0.6 \times 0.7 = 0.42$. Similarly, we can calculate the unconditional probabilities for all remaining prices.



The expected stock price is $0.42 \times 100 + 0.18 \times 90 + 0.08 \times 80 + 0.32 \times 70 = 87$

Covariance & correlation

Covariance is a measure of how two variables move together. A **positive covariance** indicates that the variables tend to move together in the same direction. Whereas, a **negative covariance** indicates that the variables tend to move in opposite directions. If two variables X and Y have expected values of $E(X)$ and $E(Y)$, then the covariance can be calculated as:

$$\text{Cov}(X,Y) = E[X - E(X)] [Y - E(Y)]$$

Calculate the covariance of two stocks A and B given two possible states of the economy. Refer to the table below.

Scenario	P(Scenario)	Expected Returns of A	Expected Returns of B
Recession	0.2	1%	3%
Expansion	0.8	8%	6%

Solution:

The expected return of A is: $0.2 \times 1 + 0.8 \times 8 = 6.6\%$

The expected return of B is: $0.2 \times 3 + 0.8 \times 6 = 5.4\%$

The covariance can be calculated as:

$$0.2 (1 - 6.6) (3 - 5.4) + 0.8 (8 - 6.6) (6 - 5.4) = 2.688 + 0.672 = 3.36$$

Correlation is a standardized measure of the linear relationship between two variables. It

is obtained by dividing the covariance of two variables by the product of their standard deviations. The correlation coefficient can range from -1 to +1.

$$\text{Corr}(X,Y) = \text{Cov}(X,Y) / \sigma(X) \sigma(Y)$$

From the previous example, the covariance between Stock A and Stock B is 3.36. Calculate the correlation given that the standard deviation of Stock A is 2.8 and the standard deviation of Stock B is 1.2

Solution:

$$\text{Corr}(A,B) = 3.36 / (2.8)(1.2) = 1$$

Expected value, variance, and standard deviation of a random variable & of returns on a portfolio

Random variable

The expected value of a random variable is the probability-weighted average of the possible outcomes of the random variable. The standard deviation / variance can be found using a financial calculator.

$$E(X) = X_1P(X_1) + X_2P(X_2) + \dots + X_nP(X_n)$$

A stock's projected EPS for the upcoming year depends on the state of the economy as shown in the table below. Calculate the expected value of EPS, variance, and standard deviation.

State of Economy	Probability	EPS
Good	0.4	\$9
Average	0.5	\$6
Weak	0.1	\$1

Solution:

$$E(X) = X_1P(X_1) + X_2P(X_2) + X_3P(X_3) = 9 \times 0.4 + 6 \times 0.5 + 1 \times 0.1 = \$6.7$$

The expected value and standard deviation can also be directly found using a financial calculator. The keystrokes are given below. [2nd] [DATA], [2nd] [CLR WRK], 9 [ENTER], [↓] 40 [ENTER], [↓] 6 [ENTER], [↓] 50 [ENTER], [↓] 1 [ENTER], [↓] 10 [ENTER], [2nd] [STAT], [2nd] [SET] press repeatedly till you see 1-V. Keep pressing [↓] to see: N = 100, X = 6.7, Sx = 2.38, $\sigma_x = 2.37$.

Note: We use the population standard deviation and not the sample standard deviation because we have entered all outcomes which means that we have covered the entire population.

To calculate the population variance, we square the standard deviation.

$$\text{Population variance} = 2.37^2 = 5.6169$$

Returns on a portfolio

The expected returns and the variance of a two-asset portfolio are given by:

$$E(R_P) = w_1 E(R_1) + w_2 E(R_2)$$

$$\sigma^2(R_P) = w_1^2 \sigma_1^2(R_1) + w_2^2 \sigma_2^2(R_2) + 2w_1 w_2 \rho(R_1, R_2) \sigma(R_1) \sigma(R_2)$$

A portfolio consists of 70% stocks and 30% bonds. The expected return on stocks is 10% and the expected return on bonds is 5%. The standard deviation of stock returns is 0.3 and the standard deviation of bond returns is 0.1. The correlation between stock returns and bond returns is 0.2. Calculate the expected return and the variance of the portfolio.

Solution:

$$\text{Expected return} = 0.7 \times 10 + 0.3 \times 5 = 8.5\%$$

$$\text{Variance} = 0.7^2 \times 0.3^2 + 0.3^2 \times 0.1^2 + 2 \times 0.7 \times 0.3 \times 0.2 \times 0.3 \times 0.1 = 0.04752$$

Calculating covariance from a joint probability function

Calculate the covariance between the returns of Stock A and Stock B, given the following joint probability table.

	$R_A = 1\%$	$R_A = 8\%$
$R_B = 3\%$	0.2	0
$R_B = 6\%$	0	0.8

Solution:

$$\text{The expected return of A is: } 0.2 \times 1 + 0.8 \times 8 = 6.6\%$$

$$\text{The expected return of B is: } 0.2 \times 3 + 0.8 \times 6 = 5.4\%$$

The covariance can be calculated as:

$$0.2(1 - 6.6)(3 - 5.4) + 0.8(8 - 6.6)(6 - 5.4) = 2.688 + 0.672 = 3.36$$

Bayes' formula

Bayes' formula is used to update the probability of an event based on new information. The formula for calculating the updated probability is:

$$P(\text{Event}|\text{Information}) = \frac{P(\text{Information}|\text{Event})}{P(\text{Information})} \times P(\text{Event})$$

The probability of a recession is 0.4 and the probability of an expansion is 0.6. If there is an expansion, the probability that the stock price will increase is 0.8. If there is a recession, the probability that the stock price will increase is 0.3. It turns out that the stock price did increase, what is the probability that we are in a recession?

Solution:

$$P(E): \text{The unconditional probability of recession} = 0.4$$

$$P(E^c): \text{The unconditional probability of an expansion} = 0.6$$

$$P(I|E): \text{The probability of increase in stock price given that we are in a recession} = 0.3$$

$$P(I|E^c): \text{The probability of increase in stock price given that we are in an expansion} = 0.8$$

$P(I)$: The unconditional probability of an increase in stock price. This can be calculated using total probability rule.

$$P(I) = P(I|E) \times P(E) + P(I|E^c) \times P(E^c)$$

$$= 0.3 \times 0.4 + 0.8 \times 0.6 = 0.6$$

Using Bayes' formula, the probability that we are in a recession given that the stock price increased is:

$$P(E|I) = P(I|E) \times P(E) / P(I)$$

$$= 0.3 \times 0.4 / 0.6 = 0.2$$

Principles of counting

Permutations is the number of ways to choose r objects from a total of n objects when the order in which the r objects are chosen is important.

$${}_n P_r = \frac{n!}{(n-r)!}$$

A portfolio manager wants to sell 4 stocks from a portfolio that consist of 10 stocks. In how many ways can the 4 stocks be chosen, when the order of sale is important?

Solution:

$${}_{10} P_4 = \frac{10!}{(10-4)!} = 5,040$$

Note: On the exam use the calculator function. The key strokes are:

10 [2nd] [-] 4 [=] 5,040

Combinations is the number of ways to choose r objects from a total of n objects when the order in which the r objects are chosen is not important.

$${}_n C_r = \binom{n}{r} = \frac{n!}{(n-r)!r!}$$

A portfolio manager wants to sell 4 stocks from a portfolio that consists of 10 stocks. In how many ways can the 4 stocks be chosen, when the order of sale is not important?

Solution:

$${}_{10} C_4 = \binom{10}{4} = \frac{10!}{(10-4)!4!} = 210$$

Note: On the exam use the calculator function. The key strokes are:

10 [2nd] [+] 4 [=] 210